Problems on Counting Techniques

A committee of 3 members is to be formed consisting of one representative each from labor, management, and the public. If there are 2 possible representatives from labor, 2 from management, and 4 from the public, determine how many different committees can be formed.

Solution

A labor representative can be chosen in 3 different ways, a management representative can be chosen in 2 different ways, and a public representative can be chosen in 4 different ways.
Therefore, the number of different committees that can be formed is

$$3\times2\times4 = 24 \text{ ways.}$$

In how many ways can 5 differently colored balls can be arranged in a row?

Solution

The first position can be occupied by any one of the 5 balls; i.e., there are five ways of filling the first position. Now, there are only 4 ways of filling the second position ... etc
Therefore the number of arrangements of 5 balls in a row is

$$5\times4\times3\times2\times1 = 120.$$

In how many ways can 10 people be seated on a bench if only 4 seats are available?

Solution

The first seat can be filled in one of 10 ways. When this happens, there will be 9 ways of filling the second seat, 8 ways of filling the third seat, and 7 ways of filling the fourth seat. Therefore the number of arrangements of 10 people taken 4 at a time is

$$= 10\times9\times8\times7$$

OR

$$P_4^{10} = \frac{10!}{(10-4)!}$$

It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

Solution

The first seat can be filled by men in one of 5 ways. The third seat can be filled by men in 4 ways …etc. . The second seat can be filled by women in one of four ways. The fourth seat can be filled by women in one of three ways ….etc

Therefore, the men can be seated in $_5P^5$ ways, while the women can be seated in $_4P^4$ ways.

►► *Number of arrangements = $_5P^5 \cdot {}_4P^4 = 5!.4! = 2880$*

How many 4-digit numbers can be formed with the 10 digits 0, 1, 2, …, 9 if
(a) repetitions are allowed
(b) repetitions are not allowed
(c) the last digit must be zero and repetitions are not allowed

<u>Solution</u>

(a) *The first digit can take any number from 1 to 9 (cannot take the zero); i.e., only 9 choices*
The second, third, and fourth digits can take any number from 0 to 9; i.e., 10 choices each.

i.e. $9\times10\times10\times10 = 9000$ *numbers can be formed*

(b) *The first digit can be any number from 1 to 9 (cannot take zero); i.e., only 9 options*
The second digit can be any of 9 (any but the number used in the first digit)
The third digit can be any of 8 (any but those used in the first two digits)
The fourth digit can be any of 7 (any but those used in the first three digits)

i.e. $9\times9\times8\times7 = 4536$ *numbers can be formed*

OR *we can say $9\times{}_3P^9$ since the 1st digit can be any of 9 while the last three digits can be chosen in $_9P^3$ ways*

(c) *The first* digit can be filled in 9 ways, the second in 8 ways, the third in 7 ways, and the fourth in one way (it must be zero as requested in the problem)

i.e. $9\times8\times7 = 504$ *numbers can be formed*

OR $9\times{}_8P^2 = 9\times8\times7 = 504$ *numbers can be formed (since the 2nd and 3rd digits can be chosen in $_8P^2$ ways)*

Five different science books, six different history books, and two different statistics books are to be arranged on a shelf. How many different arrangements are possible if
(a) The books in each particular subject must all stand together,
(b) Only the science books must stand together?

<u>Solution</u>

(a)

- The science books can be arranged among themselves in $_5P^5$ ways; i.e., 5!
- The history books can be arranged among themselves in $_6P^6$ ways; i.e., 6!
- The statistics books can be arranged among themselves in $_2P^2$ ways; i.e, 2!
- The three groups (as whole groups) can be arranged in $_3P^3$ ways; i.e., 3!
Therefore, the number of possible arrangements is

$$5!\times6!\times2!\times3! = 1,036,800$$

(b)

Consider the science books are one big book. The remaining 8 books can be arranged in $_8P^8$ ways (8!). Also, the science books can be arranged among themselves in $_5P^5$ ways.
Therefore, the number of possible arrangements is

$$8!\times5! = 4,838,400$$

Five red balls, two white balls, and three blue balls are arranged in a row. How many different arrangements are possible, knowing that all the balls of the same color are not distinguishable from each other?

Solution

If the balls of the same color are distinguishable, then the number of ways to arrange balls is $_{10}P^{10} = 10!$. However, if the balls of the same color are not distinguishable then we must divide by the number of ways in which each group can be arranged among themselves

$$N = 10!/(5!.2!.3!)$$

From the previous problem, we can conclude*:*
The number of different arrangements of n objects of which n_1 are alike, n_2 are alike, …, n_k are alike is

$$\frac{n!}{n_1!\,n_2!\ldots n_k!} \qquad \text{where } n_1 + n_2 + \ldots + n_k = n$$

In how many ways can 7 people be seated at a round table if
(a) they can sit anywhere,
(b) two particular people must sit next to each other?
(c) two particular people must not sit next to each other?

Solution

- ***a)*** *If one of them be seated anywhere. Then the remaining 6 people can be seated in 6! Ways; i.e. in 720 ways.*
- ***b)*** *Consider the 2 particular people as 1 person. Then the remaining 5 people can be arranged in 5! ways. The 2 particular people can be arranged in 2! Ways. Therefore, the number of ways of arranging 7*

people at a round table with 2 particular people sitting together = 5!.2! = 240

c) *Using (a) and (b), the total number of ways in which 7 people can be seated at a round table so that the 2 particular people do not sit together = 720 – 240 = 480 ways*